# Higher order Wess-Zumino consistency conditions in gauge theories

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# Contents

# Introduction

**Symmetries:**

$$\text{Classical Action} \quad \overset{?}{\not\to} \quad \text{Quantum Action}$$

$$\text{Classical Symmetries} \quad \overset{?}{\not\to} \quad \text{Ward Identities}$$

**Symmetries breaking:**

$$\left| \begin{array}{l} \mathit{Anomalies} \\ UV\ \mathit{Divergences} \end{array} \right. \quad \Rightarrow \quad \begin{array}{l} \mathit{Symmetries} \\ \mathit{Breaking} \end{array}$$

$$\text{Anomalies} \quad \equiv \quad \text{Truly } W\,I \text{ breaking}$$

$$\text{Counter-Terms} \quad \equiv \quad \text{Spurious } W\,I \text{ breaking}$$

**B R S T Symmetry:** $\Rightarrow$ Slavnov-Taylor Identities

$$\left\{ \begin{array}{l} B\,R\,S\,T\ \mathit{nilpotency} \\ S\,T\ \mathit{identities} \end{array} \right. \quad \Rightarrow \quad \mathit{Unitarity}$$

# Field Antifield Formalism

For each field $\Phi_a$ one introduces an antifield $\Phi_a^*$

Def antibracket:

$$(X,Y) = \sum_a \left( \frac{\delta_r X}{\delta \Phi_a} \frac{\delta_l Y}{\delta \Phi_a^*} - \frac{\delta_r X}{\delta \Phi_a^*} \frac{\delta_l Y}{\delta \Phi_a} \right)$$

Graded Jacoby Identity:

$$(X,(X,X)) = 0$$

Master Equation for the classical action:

$$\mathcal{S}\left(\Gamma^{(0)}\right) = \left(\Gamma^{(0)}, \Gamma^{(0)}\right) = 0$$

Power-counting

$\Rightarrow$ **Quantum Action Principle**

Equation for the quantum action:

$$\mathcal{S}(\Gamma) = (\Gamma, \Gamma) \quad = \quad \Delta \cdot \Gamma$$

local insertion

# Algebraic Renormalization

Green's functions $\to$ power series in $\hbar$

$$\Gamma[\Phi, \Phi^*] = \sum_{j=0}^{\infty} \hbar^j \Gamma^{(j)}[\Phi, \Phi^*]$$

$$\mathcal{S}(\Gamma) = (\Gamma, \Gamma) = \Delta \cdot \Gamma = \quad \hbar\ \mathcal{A}\ + O\left(\hbar^{(2)}\right)$$

Wess Zumino Consistency Condition:

$$\left(\Gamma^{(0)}, \mathcal{A} \quad\right) = 0\,,$$

$$\mathcal{A} \quad = \int d^4x \begin{pmatrix} \text{Lorentz-invariant } (FP = 1) \\ \text{local polynomial in} \\ \Phi, \Phi^* \text{ and their derivatives} \\ \text{of dimension } \leq d. \end{pmatrix}$$

$d = 4$

$$\Delta \cdot \Gamma \quad \sum_{j \geq 1} {}^{j} \quad {}^{j} \quad ///////$$

If $\mathcal{A}^{(j)} = 0$, $j = 0, 1, \ldots, n-1$, then

n-loop

$(n)$

$(n)$

$$+ \max_i \{4 - d_{\Phi_i} - d_{\Phi_i^*}\}$$

# Constructing $\Delta \cdot \Gamma$

First order:

$$\left(\Gamma^{(0)}, \mathcal{A}^{(1)}\right) = 0$$

$$\Rightarrow \mathcal{A}^{(1)} = \sum_{k \geq 1} \lambda_k \mathcal{C}_k + \left(\Gamma^{(0)}, \mathcal{C}_0\right),$$

$\mathcal{C}_k$ representatives of the independent non-trivial $\left(\Gamma^{(0)}, \cdot\right)$ cohomology classes with $FP$-charge $+1$.

N-th order:

- Recursive assumption

$$\mathcal{S}(\Gamma) = (\Gamma, \Gamma)^{(j)} = 0, \quad j = 0, 1, \ldots, n-1.$$

$\Rightarrow$ WZ consistency condition for $\mathcal{A}^{(n)}$

$$\left(\Gamma^{(0)}, \mathcal{A}^{(n)}\right) = 0$$

$\Rightarrow$ Solution

$$\mathcal{A}^{(n)} = \sum_{k \geq 1} \lambda_k^{(n)} \mathcal{C}_k + \left(\Gamma^{(0)}, {\mathcal{C}_0}^{(n)}\right)$$

# WZ consistency conditions to all orders

- **Recursive construction of** $\Delta \cdot \Gamma$: one must recover the master equation at lower orders before studying the $n$-th order master equation breaking term.

$\Rightarrow$ **Dependence on the normalization conditions**: even in the absence of cohomological obstructions to the restoration of the master equation, CTs must be chosen in $\Gamma^{(n)}$ so that $\left(\Gamma^{(0)}, \mathcal{C}_0{}^{(n)}\right) = 0$.

- Is it possible to construct $\Delta \cdot \Gamma$ by using cohomological arguments without recourse to the iterative assumption?

$$\mathcal{S}\left(\Gamma\right)^{(<n)} \neq 0 \Rightarrow \Delta \cdot \Gamma^{(n)} = ?$$

# Abelian Higgs-Kibble model

BRST Transformations:

$$
\begin{array}{lcl}
s\phi_1 & = & -ec\phi_2 \\
s\psi & = & -i\frac{e}{2}\gamma_5\psi c \\
sc & = & 0 \\
sA_\mu & = & \partial_\mu c
\end{array}
\qquad
\begin{array}{lcl}
s\phi_1 & = & ec(\phi_1+v) \\
s\bar\psi & = & i\frac{e}{2}c\bar\psi\gamma_5 \\
s\bar c & = & \partial A + \frac{ev}{\alpha}\phi_2 \\
s\,s\cdot & \neq & 0 \ \text{(on-shell)}
\end{array}
$$

Classical Action $\Gamma^{(0)}$:

$$
\begin{aligned}
&\int d^4x \Big[ -\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2v^2}{2}A_\mu^2 - \frac{\alpha}{2}\partial A^2 + \alpha\bar c \Box c \\
&+e^2 v(v+\phi_1)\bar c c + eA_\mu\left(\phi_2\partial^\mu\phi_1 - \partial^\mu\phi_2\phi_1\right) \\
&+\frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right) - \frac{e^2v^2}{2\alpha}\phi_2^2 \\
&+\left(\frac{e^2}{2}A^2 - \frac{\lambda}{4}\right)\left((v+\phi_1)^2+\phi_2^2\right)^2 \\
&+\bar\psi\left(i\,\partial\!\!\!/ + \frac{e}{2}\gamma_\mu\gamma_5 A^\mu + G\left(v+\phi_1 - i\gamma_5\phi_2\right)\right)\psi \\
&+J_1(-ec\phi_2) + J_2 ec(\phi_1+v) + i\frac{e}{2}\bar\eta\gamma_5\psi c \\
&+i\frac{e}{2}c\bar\psi\gamma_5\eta + \frac{M^2}{2}A_\mu^2 + M^2\bar c c - \frac{M^2}{2\alpha}(\phi_1^2+\phi_2^2)\Big]
\end{aligned}
$$

# External Sources Formalism

Slavnov-Taylor operator $\mathcal{S}(\Gamma)$:

$$\int d^4x \left[\partial^\mu c \frac{\delta\Gamma}{\delta A^\mu} + \left(\partial A + \frac{ev}{\alpha}\phi_2\right)\frac{\delta\Gamma}{\delta\bar{c}}\right] + \frac{1}{2}(\Gamma,\Gamma)$$

Def reduced antibracket:

$$(X,Y) = \sum_A \left(\frac{\delta_r X}{\delta\Phi_A}\frac{\delta_l Y}{\delta J_A} - \frac{\delta_r X}{\delta J_A}\frac{\delta_l Y}{\delta\Phi_A}\right)$$

with $\Phi_A \in \{\phi_1, \phi_2, \psi, \bar{\psi}\}$ and $J_A \in \{J_1, J_2, \bar{\eta}, \eta\}$

Linearized Slavnov-Taylor operator $\mathcal{S}_\Gamma(\cdot)$:

$$\int d^4x \left[\partial^\mu c \frac{\delta\cdot}{\delta A^\mu} + \left(\partial A + \frac{ev}{\alpha}\phi_2\right)\frac{\delta\cdot}{\delta\bar{c}}\right] + (\Gamma,\cdot)$$

Ghost equation:

$$\left(\frac{\delta\cdot}{\delta\bar{c}} - ev\frac{\delta\cdot}{\delta J_2}\right)\Gamma = \alpha\Box c + M^2 c$$

# Consistency conditions

**Massless case:**

$$\left\{\begin{array}{l} \mathcal{S}_{\Gamma}\left(\mathcal{S}\left(\Gamma\right)\right)=0 \\ \mathcal{S}\left(\Gamma^{(0)}\right)=0 \end{array}\right. \quad \Rightarrow \quad \mathcal{S}^2_{\Gamma^{(0)}}=0$$

1-st order:

$$\left\{\begin{array}{l} \mathcal{S}_{\Gamma^{(0)}}\left(\mathcal{S}\left(\Gamma\right)^{(1)}\right)=0 \\ \text{nilpotency of } \mathcal{S}_{\Gamma^{(0)}} \end{array}\right. \quad \Rightarrow \quad \begin{array}{r} \mathcal{S}\left(\Gamma\right)^{(1)}=\mathcal{Y}^{(1)} \\ +\mathcal{S}_{\Gamma^{(0)}}\left({\mathcal{C}_0}^{(1)}\right) \end{array}$$

**Massive case:**

$$\mathcal{S}_{\Gamma}\left(\mathcal{S}\left(\Gamma\right)\right)=-\frac{M^2}{\alpha}\int d^4x\, c\frac{\delta\Gamma}{\delta\bar{c}}$$

$$\mathcal{S}\left(\Gamma^{(0)}\right)=0 \quad \text{and} \quad \mathcal{S}^2_{\Gamma^{(0)}}=-\frac{M^2}{\alpha}\int d^4x\, c\frac{\delta}{\delta\bar{c}}$$

$$\text{but} \quad \mathcal{S}^2_{\Gamma^{(0)}}\left(action-like\right)=0$$

1-st order:

$$\mathcal{S}_{\Gamma^{(0)}}\left(\mathcal{S}\left(\Gamma\right)^{(1)}\right)=-\frac{M^2}{\alpha}\int d^4x\, c\frac{\delta\Gamma}{\delta\bar{c}} \quad =0 \quad \begin{array}{c}\text{GE}\\ \text{QAP}\end{array}$$

# Consistency conditions: Higher orders

$$\mathcal{S}_\Gamma\left(\mathcal{S}\left(\Gamma\right)\right) = -\frac{M^2}{\alpha}\int d^4x\, c\frac{\delta\Gamma}{\delta\bar{c}}$$

2-nd order:

$$\begin{aligned}
\mathcal{S}_{\Gamma^{(0)}}\left(\mathcal{S}\left(\Gamma\right)\right)^{(2)} \quad &+ \quad \mathcal{S}_{\Gamma^{(1)}}\left(\mathcal{S}\left(\Gamma\right)\right)^{(1)} \\
&= \quad -\frac{M^2}{\alpha}\int d^4x\, c\frac{\delta\Gamma^{(2)}}{\delta\bar{c}} = 0 \ (QAP)
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}\left(\Gamma\right)^{(1)} = 0 \quad &\Rightarrow \quad \mathcal{S}_{\Gamma^{(0)}}\left(\mathcal{S}\left(\Gamma\right)^{(2)}\right) = 0 \\
&\Rightarrow \quad \mathcal{S}\left(\Gamma\right)^{(2)} = \mathcal{Y}^{(2)} + \mathcal{S}_{\Gamma^{(0)}}\left({\mathcal{C}_0}^{(2)}\right) \\
&\qquad \text{with } \mathcal{Y}^{(2)} \text{ and } {\mathcal{C}_0}^{(2)} \text{ local}
\end{aligned}$$

$$\mathcal{S}\left(\Gamma\right)^{(1)} \neq 0 \quad \Rightarrow \quad \mathcal{S}\left(\Gamma\right)^{(2)} = ?$$

# Consistency conditions: an explicit example

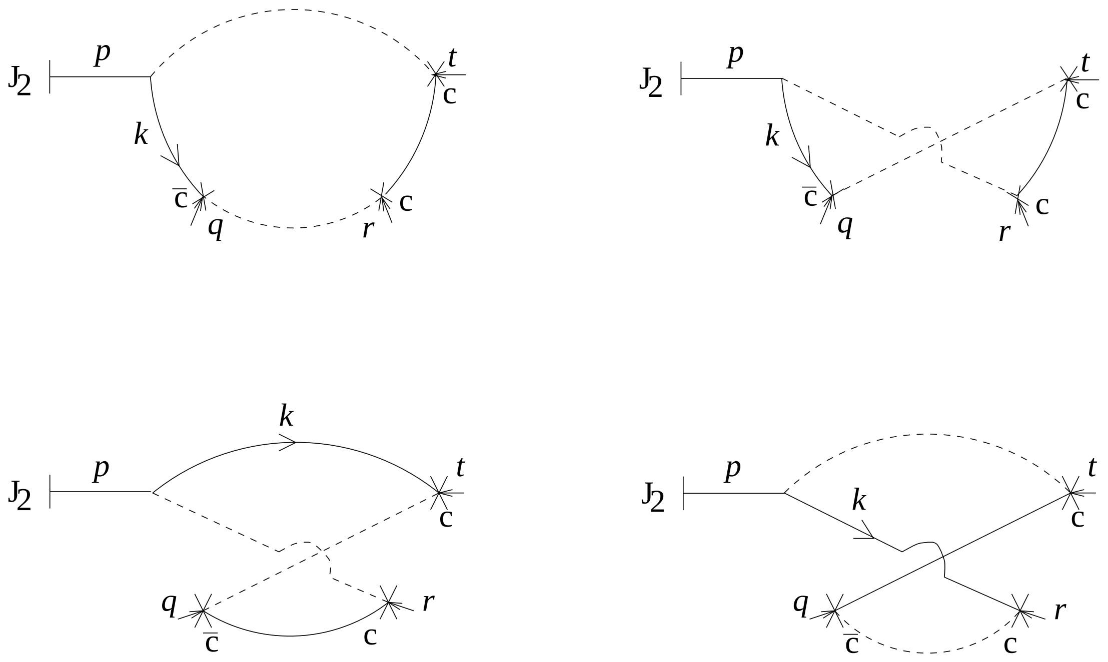
$$\mathcal{P}(\cdot) \equiv \frac{\delta^4(\cdot)}{\delta\bar{c}(q)\delta c(s)\delta c(t)\delta c(r)}$$

$$X \equiv \mathcal{S}\left(\Gamma\right)^{(2)}$$

$$\mathcal{S}_{\Gamma^{(1)}}\left(\mathcal{S}\left(\Gamma\right)^{(1)}\right) = -\sum_i \int d^4x \, \frac{\delta\Gamma^{(1)}}{\delta J_i(x)} \frac{\delta\mathcal{S}\left(\Gamma\right)^{(1)}}{\delta\phi_i(x)}$$

$$\mathcal{P}\mathcal{S}_{\Gamma^{(0)}}\left(\mathcal{S}\left(\Gamma\right)^{(2)}\right)\Big|_{\varphi=0} = -\mathcal{P}\mathcal{S}_{\Gamma^{(1)}}\left(\mathcal{S}\left(\Gamma\right)^{(1)}\right)\Big|_{\varphi=0}$$

$$\begin{aligned}
&\big(-ir_\mu X_{\bar{c}cc A_\mu} &&+ evX_{\bar{c}cc\phi_2} + e^2 v X_{ccJ_1}\big)\big|_{\varphi=0} \\
& &&+\text{cycl. perm. of momenta} \\
&= \Big(\mathcal{S}\left(\Gamma\right)^{(1)}_{c\phi_2} &&\Gamma^{(1)}_{\bar{c}ccJ_2}\Big)\Big|_{\varphi=0} \\
& &&+\text{cycl. perm. of momenta}
\end{aligned}$$




Graphs contributing to $\left.\Gamma^{(1)}_{\bar{c}(q)c(r)c(t)J_2(p)}\right|_{\varphi=0}$

$\left.\Gamma^{(1)}_{\bar{c}(q)c(r)c(t)J_2(p)}\right|_{\varphi=0}$ not local

$$\left.\mathcal{S}\left(\Gamma\right)^{(1)}_{c(s)\phi_2(p)}\right|_{\varphi=0} = \left(a + bs^2\right)\delta^{(4)}(s+p)$$

$\Rightarrow X_{\bar{c}cA_\mu}, X_{\bar{c}c\phi_2}, X_{ccJ_1}$

non-polynomial in the external momenta

# Conclusions

Which requirements are needed for the first order Wess-Zumino consistency condition to hold true at the next order in perturbation theory?

**a)** If the cohomologically trivial contributions to the first order ST breaking terms are not removed by a suitable choice of the first order normalization conditions, the Wess-Zumino consistency condition is modified at the second order.

**b)** Moreover, if one fails to recover the cohomologically trivial part of the first order STI breaking local functional, the second order anomaly actually turns out to be a non-local functional of the fields and the external sources.

# References


[1] P.Piguet and S.P.Sorella, Lecture Notes in Physics m28, Springer-Verlag (1995). Y.M.P.Lam, *Phys. Rev.* **D6** (1972) 2145, *Phys. Rev.* **D7** (1973) 2943; J.Lowenstein, *Comm. Math. Phys.* **24** (1971) 1; P.Breitenlohner and D.Maison, *Comm. Math. Phys.* **52** (1977) 11, 39, 55.

[2] C.Becchi, A.Rouet and R.Stora, *Phys. Lett.* **B32** (1974) 344; L.V.Tyutin, Lebedev preprint FIAN n.39 (1975). G.Curci and R.Ferrari, *Il Nuovo Cimento* **35A** (1976), 273; T.Kugo and I.Ojima, *Phys. Lett.* **B73** (1978) 459.

[3] J.Gomis, J.Paris, S.Samuel, *Phys. Rep.* **259** (1995), 1-145.

[4] J.Zinn-Justin, Springer-Verlag, Berlin 1975. J.Wess, B.Zumino, *Phys. Lett.* **B37** (1971), 95.

[5] P.L.White, *Phys .Lett.* **B284** (1992) 55-60. J.Paris, W.Troost, *Nucl. Phys.* **B482** (1996) 373-402. M.Picariello and A.Quadri, hep-th/0001174

[6] C.Becchi, A.Rouet and R.Stora, *Comm. Math. Phys.* **42** (1975) 127. R.Ferrari and P.A.Grassi, *Phys. Rev.* **D60**:065010 (1999). R.Ferrari, P.A.Grassi and A.Quadri, *Phys. Lett.* **B**472 (2000) 346-356.