The Equivalence Theorem: Renormalizability of Nonrenormalizable Theories?

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Introduction

Classical Action $S \rightarrow$ Quantum Action Γ

Path integral and Field redefinition:

$$Z[J] \propto \int \mathcal{D}\varphi e^{iS_0[\varphi]}$$

$$\Downarrow$$

φ is a dumpy variable

$$\Downarrow$$

$\varphi \rightarrow \Phi(\varphi, \rho)$ should be a good redefinition

$$\Downarrow$$

$$Z[J] \propto \int \mathcal{D}\varphi e^{iS_0[\Phi(\varphi,\rho)]+D(\varphi,\rho)}$$

$+i \int_x J\varphi$

$\sim$

$+i \int_x J\varphi$

Strong Equivalence Theorem

$$+i \int_x J\varphi$$

$$+i \int_x J\Phi(\varphi,\rho)$$

Weak Equivalence Theorem

Weak Equivalence Theorem

The S-matrix obtained from

$$Z[J] \propto \int \mathcal{D}\varphi e^{iS_0[\varphi]+i\int_x J\varphi}$$

is the same as the one obtained from

$$Z[J] \propto \int \mathcal{D}\varphi e^{iS_0[\Phi(\varphi,\rho)]+D(\varphi,\rho)+i\int_x J\Phi(\varphi,\rho)}$$

Strong Equivalence Theorem

The following equalities hold true:

$$\int \mathcal{D}\Phi\Big(\prod_{i=1}^{n}\mathcal{O}_i(\Phi(x_i))\Big)e^{iS_0[\Phi]}$$

$$\|$$

$$\int \mathcal{D}\varphi\Big(\prod_{i=1}^{n}\mathcal{O}_i(\Phi(\varphi,\rho)(x_i))\Big)e^{iS_0[\Phi(\varphi,\rho)]+D(\varphi,\rho)}$$

Classical Formalism

$$Z[J] \quad \propto \int [\mathcal{D}\varphi][\mathcal{D}\chi][\mathcal{D}B][\mathcal{D}\bar{c}][\mathcal{D}c]$$

$$e^{iS_0[\chi]}$$

$$+\int_x B(\chi-\Phi(\varphi,\rho))+\int_x \bar{c}\quad\frac{\delta\Phi}{\delta\varphi}c$$

$$+\int_x B(\chi-\Phi(\varphi,\rho))+\int_x \bar{c}\left(\frac{\delta\Phi}{\delta\varphi}c+\frac{\delta\Phi}{\delta\rho}\psi\right)$$

Notice:

$$det\mathcal{J} = \int [\mathcal{D}\bar{c}][\mathcal{D}c]e^{\int_x \bar{c}\mathcal{J}c} = \int [\mathcal{D}\bar{c}][\mathcal{D}c]e^{\int_x \bar{c}(\mathcal{J}c+\xi)}$$

Under the following transformations:

$$s\varphi = c\,, \quad s\bar{c} = B\,, \quad s\rho = \psi\,, \quad s\chi = 0\,,$$
$$sc = 0\,, \quad sB = 0\,, \quad s\psi = 0$$

$$Z[J] \propto \int [\mathcal{D}\varphi][\mathcal{D}\chi][\mathcal{D}B][\mathcal{D}\bar{c}][\mathcal{D}c]e^{iS_0[\chi]+s\int_x \bar{c}(\chi-\Phi(\varphi,\rho))}$$

At classical level:

$$\mathcal{S}\left(\Gamma^{(0)}\right) \equiv \int_x c\frac{\delta\Gamma^{(0)}}{\delta\varphi} + \int_x B\frac{\delta\Gamma^{(0)}}{\delta\bar{c}} + \psi\frac{\partial\Gamma^{(0)}}{\partial\rho} = 0$$

The locally invertible case

Classical level

$\Phi(\varphi, \rho)$ locally invertible

$$\Downarrow$$

Eq. of Motion for c $\quad \Rightarrow \quad \frac{\delta \Phi}{\delta \varphi} c + \frac{\partial \Phi}{\partial \rho} \psi = 0$

$$\Updownarrow$$

$$s\varphi = c = -\left(\frac{\delta \Phi}{\delta \varphi} \right)^{-1} \frac{\partial \Phi}{\partial \rho} \psi$$

We end up with:

$$\tilde{\Gamma}^{(0)}[\varphi, \rho, \psi, \varphi^*, \beta] = S_0[\Phi(\varphi, \rho), \beta] + \int_x \varphi^* s\varphi$$

and at classical level:

$$\tilde{\mathcal{S}}(\tilde{\Gamma}^{(0)}) \equiv \int_x \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \varphi^*} \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \varphi} + \psi \frac{\partial \tilde{\Gamma}^{(0)}}{\partial \rho} = 0$$

The Quantum Action Principle

Proposition 1 *Let Γ be the vertex functional corresponding to a power-counting renormalizable theory with a classical action given by*

$$\Gamma^{(0)} = \int_x \mathcal{L}(\phi_a, \rho_i, \lambda)$$

where ϕ_a=fields, ρ_i=sources.
Then

Equation of Motion

$$\frac{\delta\Gamma}{\delta\phi_a} = \Delta_a \cdot \Gamma = \hbar^n \Delta_a + O(\hbar^{n+1})$$

Non linear variation

$$\frac{\delta\Gamma}{\delta\rho_i}\frac{\delta\Gamma}{\delta\phi_A} = \Delta_{ai} \cdot \Gamma = \hbar^n \Delta_{ai} + O(\hbar^{n+1})$$

where Δ are local operator of bounded dimensions.

The Quasi Classical Action Principle

Proposition 2 *Let Γ be the vertex functional corresponding to a theory with a classical action given by*

$$\Gamma^{(0)} = \int_x \mathcal{L}(\phi_a, \rho_i, \lambda)$$

where ϕ_a=fields, ρ_i=sources.
Then

Equation of Motion

$$\frac{\delta\Gamma}{\delta\phi_a} = \hbar^n \Delta_a + O(\hbar^{n+1})$$

Non linear variation

$$\frac{\delta\Gamma}{\delta\rho_i}\frac{\delta\Gamma}{\delta\phi_A} = \hbar^n \Delta_{ai} + O(\hbar^{n+1})$$

where Δ are local operator (formal power series).

The locally invertible case

Quantum extension

Lemma 1 *Given the classical action*

$$\tilde{\Gamma}^{(0)}[\varphi, \rho, \psi, \varphi^*, \beta] = S_0[\Phi(\varphi, \rho), \beta] + \int_x \varphi^* s\varphi$$

then at quantum level

$$\tilde{\mathcal{S}}(\tilde{\Gamma}) \equiv \int_x \frac{\delta\tilde{\Gamma}}{\delta\varphi^*}\frac{\delta\tilde{\Gamma}}{\delta\varphi} + \psi\frac{\partial\tilde{\Gamma}}{\partial\rho} = 0$$

Recursive proof:

$$\left\{\begin{array}{c} \tilde{\mathcal{S}}(\tilde{\Gamma})^{(j)} = 0\,, j < n \\ \tilde{\mathcal{S}}(\tilde{\Gamma})^{(n)} = \Delta \\ QCAP \end{array}\right. \quad \Rightarrow \quad \begin{array}{c} \Delta \text{ local power series} \\ FP(\Delta) = +1 \end{array}$$

Nilpotence: $\tilde{\mathcal{S}}^2 = 0$

$$\Downarrow$$

Wess-Zumino consistency condition:

$$\tilde{\mathcal{S}}_0(\tilde{\mathcal{S}}(\tilde{\Gamma})^{(n)}) \equiv \left(\int_x \frac{\delta\tilde{\Gamma}^{(0)}}{\delta\varphi^*}\frac{\delta}{\delta\varphi} + \psi\frac{\partial}{\partial\rho}\right)\Delta = 0$$

The locally invertible case

Quantum extension

Cohomological analyses of Δ

$$\left\{\begin{array}{c} FP(\Delta) = +1 \\ FP(\psi) = +1 \\ FP(\varphi) = 0 \end{array}\right. \quad \Rightarrow \quad \Delta|_{\psi=0} = 0$$

$$\left\{\begin{array}{c} (\rho, \psi) \text{ doublet under } \tilde{\mathcal{S}}_0 \\ \Delta|_{\psi=0} = 0 \end{array}\right. \quad \Rightarrow \quad \Delta \equiv \tilde{\mathcal{S}}_0(\Xi)$$

By adding Ξ (of order $\hbar^n$) to $\tilde{\Gamma}$ we obtain:

$$\tilde{\mathcal{S}}(\tilde{\Gamma})^{(n)} = 0$$

Q.E.D.

General case

Quantum extension

Lemma 2 *Given the classical action*

$$\begin{aligned}\Gamma^{(0)} &= S_0[\chi] + \int_x B(\chi - \Phi(\varphi, \rho)) \\ &+ \int_x \bar{c}\left(\frac{\delta\Phi}{\delta\varphi}c + \frac{\delta\Phi}{\delta\rho}\psi\right) + \int_x \beta O(\chi)\end{aligned}$$

then at quantum level

$$\mathcal{S}(\Gamma) \equiv \int_x c\frac{\delta\Gamma}{\delta\varphi} + \int_x B\frac{\delta\Gamma}{\delta\bar{c}} + \psi\frac{\partial\Gamma}{\partial\rho} = 0$$

Recursive proof:

$$\left\{\begin{array}{c}\mathcal{S}(\Gamma)^{(j)} = 0\,, j < n \\ \mathcal{S}(\Gamma)^{(n)} = \Delta \\ QCAP\end{array}\right. \quad \Rightarrow \quad \begin{array}{c}\Delta \text{ local power series} \\ FP(\Delta) = +1\end{array}$$

General case

Quantum extension

Cohomological analyses

Nilpotence: $\tilde{\mathcal{S}}_0{}^2 = 0$

$$\Downarrow$$

Wess-Zumino consistency condition:

$$\mathcal{S}_0(\mathcal{S}\,(\Gamma)^{(n)}) \equiv \left(\int_x c\frac{\delta}{\delta\varphi} + \int_x B\frac{\delta}{\delta\bar{c}} + \psi\frac{\partial}{\partial\rho}\right)\Delta = 0$$

$$\left\{\begin{array}{c} \text{One can find the homotopy} \\ \mathcal{K} \text{ such that} \\ \tilde{\mathcal{S}}_0\mathcal{K}X = X \Leftrightarrow \tilde{\mathcal{S}}_0 X = 0 \end{array}\right. \quad \Rightarrow \quad \Delta \equiv \mathcal{S}\,(\Xi)$$

By adding Ξ (of order $\hbar^n$) to Γ we obtain:

$$\mathcal{S}\,(\Gamma)^{(n)} = 0$$

Q.E.D.

Conclusions

Equivalence Theorem

How to use it

Transformation:

Locally invertible	Non Locally invertible
$\Downarrow$	$\Downarrow$
On-shell	Off-shell
$\Downarrow$	$\Downarrow$
$S_0[\Phi(\varphi,\rho)]$	$S_0[\chi] + \int_x B(\chi - \Phi(\varphi,\rho)) + \mathcal{J}$

$\Uparrow$

QAP	QCAP
$\Uparrow$	$+$
Standard S_0	*Non Standard* S_0

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