THEORY OF NEUTRINO OSCILLATIONS

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- ⇝ Critical Discussion of Standard Theory of Neutrino Oscillations
- ⇝ Quantum Mechanical Wave-Packet Approach
- ⇝ Neutrino Wave Packets in Quantum Field Theory

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Standard Theory of Neutrino Oscillations in Vacuum

[Bilenky, Pontecorvo, Phys. Rep. 41 (1978) 225]

Neutrino Production: $j_\rho^{\mathrm{CC}} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} \gamma_\rho \ell_{\alpha L}$ $\qquad \nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL}$ Fields

$$\langle 0 | \nu_{\alpha L} | \nu_\beta \rangle = \sum_{k,j} U_{\alpha k} U_{\beta j}^* \underbrace{\langle 0 | \nu_{kL} | \nu_j \rangle}_{\propto \delta_{kj}} \propto \sum_k U_{\alpha k} U_{\beta k}^* = \delta_{\alpha\beta}$$

$|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$ States

$$\mathcal{H} |\nu_k\rangle = E_k |\nu_k\rangle \quad \Rightarrow \quad |\nu_k(t)\rangle = e^{-iE_k t} |\nu_k\rangle \quad \Rightarrow \quad |\nu_\alpha(t)\rangle = \sum_k U_{\alpha k}^* e^{-iE_k t} |\nu_k\rangle$$

$$|\nu_k\rangle = \sum_{\beta=e,\mu,\tau} U_{\beta k} |\nu_\beta\rangle$$

$$|\nu_\alpha(t)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k} \right)}_{\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t)} |\nu_\beta\rangle$$

Transition Probability: $P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \left| \mathcal{A}_{\nu_\alpha \to \nu_\beta}(t) \right|^2 = \left| \sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k} \right|^2$

$$P_{\nu_\alpha \to \nu_\beta}(t) = \sum_{k,j} U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*} \exp\left[-i\left(E_k - E_j\right)t\right]$$

Relativistic Approximation + Assumption $p_k = p = E$

$$E_k = \sqrt{p^2 + m_k^2} \simeq p + \frac{m_k^2}{2p} = E + \frac{m_k^2}{2E} \quad \Longrightarrow \quad E_k - E_j \simeq \frac{\Delta m_{kj}^2}{2E}$$

$$\boxed{\Delta m_{kj}^2 \equiv m_k^2 - m_j^2}$$

Approximation $t \simeq L$ $\Longrightarrow$

$$\boxed{P_{\nu_\alpha \to \nu_\beta}(L) \simeq \sum_{k,j} U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*} \exp\left[-i \frac{\Delta m_{kj}^2 L}{2E}\right]}$$

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_{k} |U_{\alpha k}|^2 |U_{\beta k}|^2$$ $\Leftarrow$ constant term

$$+ 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*} \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$ $\Leftarrow$ oscillating term

$\Updownarrow$

COHERENCE

Main Assumptions of Standard Theory

(A1) Neutrinos are extremely relativistic particles OK!

(A2) Neutrinos produced in CC weak interaction processes together with charged leptons α^+ are described by the flavor state $|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$

Correct approximation for ultrarelativistic ν's [Giunti, Kim, Lee, PRD 45 (1992) 2414]

(A3) Massive neutrino states $|\nu_k\rangle$ have the same momentum $p_k = p$ ("Equal Momentum Assumption") and different energies: $E_k \simeq E + \frac{m_k^2}{2E}$

Unrealistic assumption, forbidden by energy-momentum conservation and Lorentz invariance, but gives correct result (as well as the "Equal Energy Assumption")

[Winter, LNC 30 (1981) 101], [Giunti, Kim, FPL 14 (2001) 213], [Giunti, MPLA 16 (2001) 2363], [Giunti, hep-ph/0302026]

(A4) Propagation Time $T \simeq L$ Source-Detector Distance OK!

⇕

WAVE PACKETS

Easy Example of Neutrino Production: $\pi^+ \to \mu^+ + \nu_\mu \quad \pi^- \to \mu^- + \bar{\nu}_\mu$

two-body decay $\Longrightarrow$ fixed kinematics

$$E_k^2 = p_k^2 + m_k^2$$

π at rest:

$$\begin{cases} p_k^2 = \dfrac{m_\pi^2}{4}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)^2 - \dfrac{m_k^2}{2}\left(1 + \dfrac{m_\mu^2}{m_\pi^2}\right) + \dfrac{m_k^4}{4\,m_\pi^2} \\[2ex] E_k^2 = \dfrac{m_\pi^2}{4}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)^2 + \dfrac{m_k^2}{2}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right) + \dfrac{m_k^4}{4\,m_\pi^2} \end{cases}$$

0^{th} order: $m_k = 0 \Rightarrow p_k = E_k = E = \dfrac{m_\pi}{2}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right) \simeq 30\,\text{MeV}$

1^{st} order: $$E_k \simeq E + \xi\,\frac{m_k^2}{2E} \qquad p_k \simeq E - (1-\xi)\,\frac{m_k^2}{2E} \qquad \xi = \frac{1}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 0.2$$

↖ ↗ general!

Plane Wave Approximation

LORENTZ INVARIANT

OSCILLATION PROBABILITY

$$P_{\nu_\alpha \to \nu_\beta}(L,T) = \left| \sum_k U_{\alpha k}^* \, e^{ip_k L - iE_k t} \, U_{\beta k} \right|^2$$

[Dolgov et al., NPB 502 (1997) 3], [Dolgov, hep-ph/0004032], [Giunti, Kim, FPL 14 (2001) 213], [Bilenky, Giunti, IJMPA 16 (2001) 3931]

$$P_{\nu_\alpha \to \nu_\beta}(L,T) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \, e^{i(p_k - p_j)L - i(E_k - E_j)T}$$

relativistic approximation: $p_k - p_j \simeq -(1-\xi)\,\frac{\Delta m^2_{kj}}{2E}$ $\quad E_k - E_j \simeq \xi\,\frac{\Delta m^2_{kj}}{2E}$

$$P_{\nu_\alpha \to \nu_\beta}(L,T) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* e^{-i(1-\xi)\frac{\Delta m^2_{kj}}{2E}L - i\xi\frac{\Delta m^2_{kj}}{2E}T}$$

$T = L \Longrightarrow$

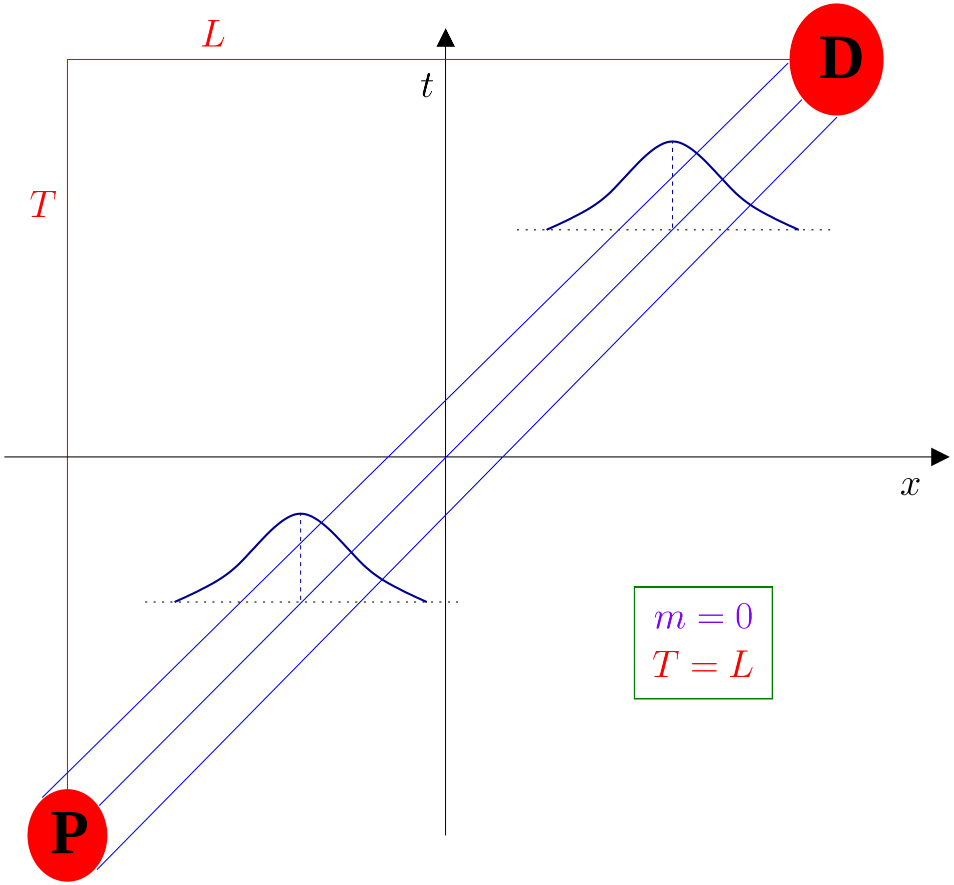
$$P_{\nu_\alpha \to \nu_\beta}(L,T) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* e^{-i\frac{\Delta m^2_{kj}}{2E}L}$$

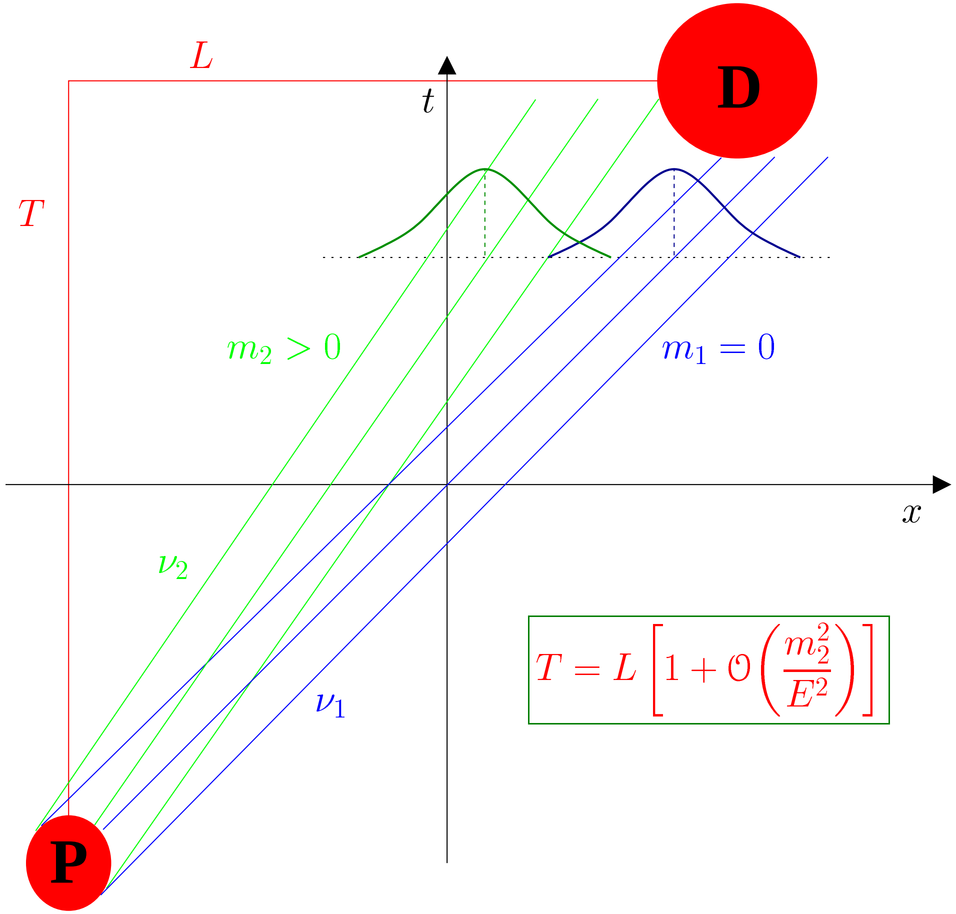
$T = L \iff$ WAVE PACKETS

Other Motivations:

[Kayser, PRD 24 (1981) 110]

- ⇝ Localization of Production and Detection Processes
- ⇝ Exact Energy-Momentum conservation would imply creation and detection of only one massive neutrino (neutrino mass measurement)





Coherence Length

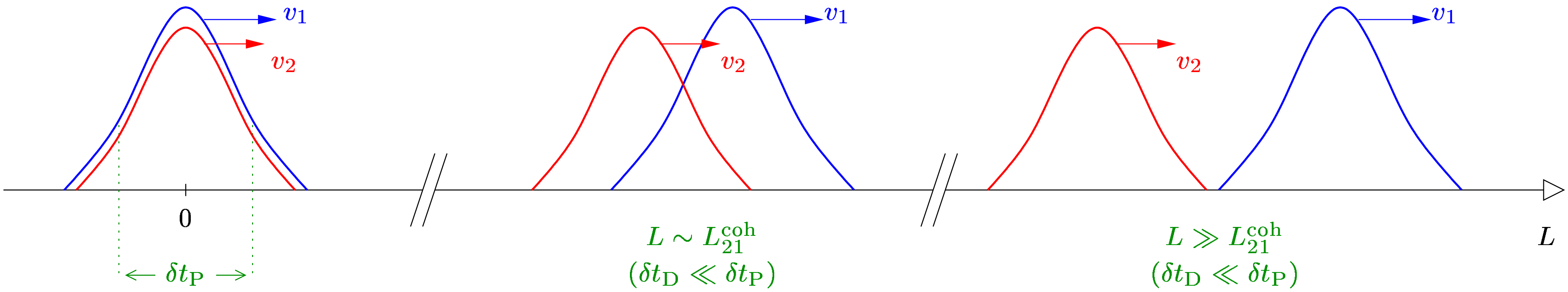
[Nussinov, PLB 63 (1976) 201], [Kiers, Nussinov, Weiss, PRD 53 (1996) 537]

Wave Packets have different velocities and separate

Different massive neutrinos can interfere if and only if wave packets arrive with $\delta t_{kj} < \delta t_{\mathrm{D}}$

$\Longrightarrow \quad L \lesssim L^{\mathrm{coh}}_{kj}$

$$|\delta t_{kj}| \simeq |v_k - v_j|\, T \simeq \frac{|\Delta m^2_{kj}|}{2E^2} L \qquad \Longrightarrow \qquad L^{\mathrm{coh}}_{kj} \sim \frac{2E^2}{|\Delta m^2_{kj}|} \sqrt{\delta t_{\mathrm{P}}^2 + \delta t_{\mathrm{D}}^2}$$



Quantum Mechanical Wave Packet Model

[Giunti, Kim, Lee, PRD 44 (1991) 3635], [Giunti, Kim, PRD 58 (1998) 017301]

Confirmed Standard Oscillation Length

Derived Coherence Length

Problems:

- ⇝ Flavor state has to be assumed:

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} \int \mathrm{d}p \, \psi_k(p) |\nu_k(p)\rangle$$

- ⇝ Neutrino properties have to be assumed:

$$E_k \simeq E + \xi \, \frac{m_k^2}{2E} \qquad p_k \simeq E - (1 - \xi) \, \frac{m_k^2}{2E}$$

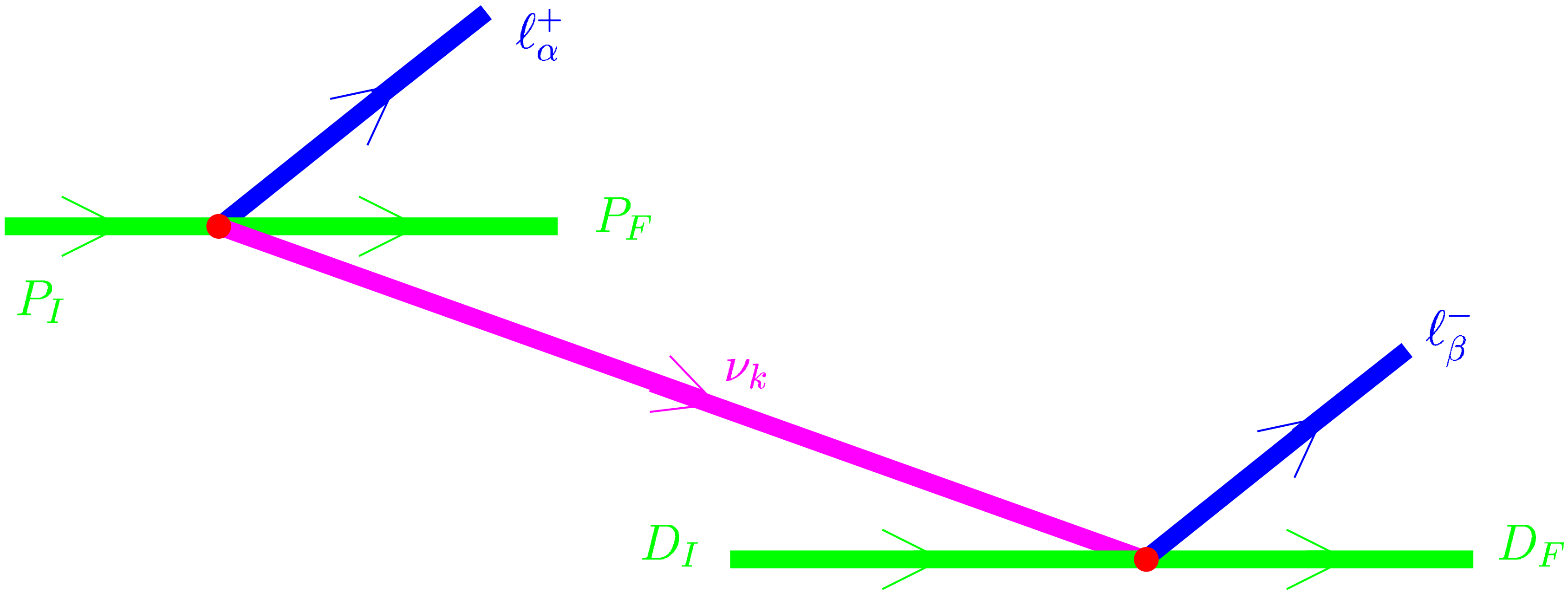
⇓

Quantum Field Theoretical Wave Packet Model

Quantum Field Theory of Neutrino Oscillations with external particles in Production and Detection processes described by wave packets and intermediate virtual neutrino

[Giunti, Kim, Lee, Lee, PRD 48 (1993) 4310]
[Giunti, Kim, Lee, PLB 421 (1998) 237]
[Cardall, PRD 61 (2000) 073006]
[Beuthe, PRD 66 (2002) 013003]
[Beuthe, PLREP 375 (2003) 105]

$$P_I \to P_F + \ell_\alpha^+ + \nu_\alpha \xrightarrow{\nu_\alpha \to \nu_\beta} \nu_\beta + D_I \to D_F + \ell_\beta^-$$



$$P_{\nu_\alpha \to \nu_\beta} \propto \left| \sum_k \left\{ P_I \to P_F + \ell_\alpha^+ + \nu_k \xrightarrow{\text{propagator}} \nu_k + D_I \to D_F + \ell_\beta^- \right\} \right|^2$$

Confirmed Standard Oscillation Length

Derived Coherence Length

“Philosophical” Problem: neutrino has no properties!

In oscillation experiments neutrinos propagate as free particles over macroscopically large distance, sometimes astronomical distances (solar, atmospheric neutrinos)

It must be possible to describe neutrinos in oscillation experiments with appropriate state, as in the quantum-mechanical approach

Neutrino Wave Packets in Quantum Field Theory

[Giunti, JHEP 11 (2002) 017]

In Quantum Field Theory $$|f\rangle \propto (\mathcal{S} - \mathbf{1})|i\rangle \simeq -i \int \mathrm{d}^4x \, \mathcal{H}_I(x) \, |i\rangle$$

Entangled Final State in Production Process: $$|\widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto -i \int \mathrm{d}^4x \, \mathcal{H}_I(x) \, |P_I\rangle$$

Disentangled by Interaction with Surrounding Medium (Measurement):

$$|\nu_\alpha\rangle \propto \langle P_F, \ell_\alpha^+ | \widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto \langle P_F, \ell_\alpha^+ | - i \int \mathrm{d}^4x \, \mathcal{H}_I(x) \, |P_I\rangle$$

Localization: $$|\chi\rangle = \int \mathrm{d}^3p \, \psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi}) |\chi(\vec{p})\rangle \qquad (\chi = P_I, P_F, \ell_\alpha^+)$$

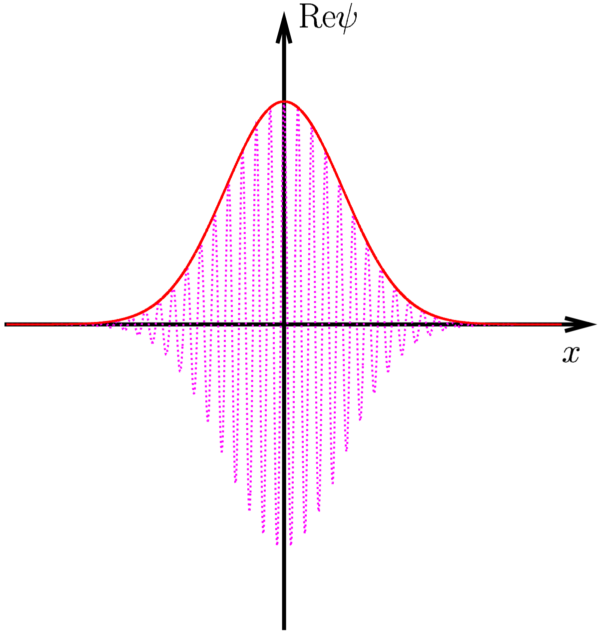
Gaussian Momentum Distribution: $$\psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi}) = \left(2\pi\sigma_{p\chi}^2\right)^{-3/4} \exp\left[-\frac{(\vec{p}-\vec{p}_\chi)^2}{4\sigma_{p\chi}^2}\right]$$

Wave Function: $$\psi_\chi(\vec{x},t;\vec{p}_\chi,\sigma_{p\chi}) = \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} \, \psi_\chi(\vec{p};\vec{p}_\chi,\sigma_{p\chi}) \, e^{-iE_\chi(\vec{p})t+i\vec{p}\vec{x}}$$

Dispersion Relation: $$E_\chi(\vec{p}) = \sqrt{\vec{p}^2 + m_\chi^2} \simeq E_\chi + \vec{v}_\chi \left(\vec{p} - \vec{p}_\chi\right)$$

Average Energy: $E_\chi = \sqrt{\vec{p}_\chi^2 + m_\chi^2}$ Group Velocity: $\vec{v}_\chi \equiv \left.\frac{\partial E_\chi}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_\chi} = \frac{\vec{p}_\chi}{E_\chi}$

Wave Function: $$\psi_\chi(\vec{x},t;\vec{p}_\chi,\sigma_{p\chi}) \simeq \left(2\pi\sigma_{x\chi}^2\right)^{-3/4} \exp\left[-iE_\chi t + i\vec{p}_\chi\cdot\vec{x} - \frac{(\vec{x}-\vec{v}_\chi t)^2}{4\sigma_{x\chi}^2}\right]$$



Space and Momentum Uncertainties: $$\sigma_{x\chi}\,\sigma_{p\chi} = \frac{1}{2}$$

Squared Energy Uncertainty: $$\langle(\delta E)^2\rangle_\chi = \langle\chi|(\widehat{E}-E_\chi)^2|\chi\rangle = \vec{v}_\chi^{\,2}\,\sigma_{p\chi}^2$$ [Beuthe, priv. comm. 2002]

State Describing Produced Neutrino:

$$|\nu_\alpha\rangle = N_\alpha \sum_k U_{\alpha k}^* \int \mathrm{d}^3p \, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p},h) \, |\nu_k(\vec{p},h)\rangle$$

$$\mathcal{A}_k^P(\vec{p},h) = \bar{u}_{\nu_k}(\vec{p},h) \, \gamma^\rho \, (1-\gamma_5) \, v_{\ell_\alpha^+}(\vec{p}_{\ell_\alpha^+}, h_{\ell_\alpha^+}) \, J_\rho^P(\vec{p}_{P_F}, \vec{p}_{P_I})$$

$e^{-S_k^P(\vec{p})}$ replaces energy-momentum δ-function

$|\nu_\alpha\rangle$ is a superposition of the massive neutrino wave packets

$$|\nu_k\rangle = N_k \int \mathrm{d}^3p \, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p},h) \, |\nu_k(\vec{p},h)\rangle$$

Squared Energy-Momentum Uncertainties of Ultrarelativistic ν_k:

$$\langle(\delta p)^2\rangle_k \sim \langle(\delta E)^2\rangle_k \sim \sigma_{pP}^2 = \sigma_{pP_I}^2 + \sigma_{pP_F}^2 + \sigma_{p\ell_\alpha^+}^2$$

Detection Process at $(\vec{L}, T)$: $|\nu_\alpha(\vec{L}, T)\rangle = e^{-i\widehat{E}T + i\widehat{\vec{P}}\cdot\vec{L}}\,|\nu_\alpha\rangle$

$$|\nu_\alpha(\vec{L}, T)\rangle = N_\alpha \sum_k U_{\alpha k}^* \int \mathrm{d}^3p\, e^{-iE_{\nu_k}(\vec{p})T + i\vec{p}\cdot\vec{L}}\, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p}, h)\, |\nu_k(\vec{p}, h)\rangle$$

Detection Amplitude: $\mathcal{A}_{\alpha\beta}(\vec{L}, T) = \langle D_F, \ell_\beta^- | -i \int \mathrm{d}^4x\, \mathcal{H}_I(x)\, |D_I, \nu_\alpha(\vec{L}, T)\rangle$

$$\mathcal{A}_{\alpha\beta}(\vec{L}, T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \sum_h \int \mathrm{d}^3p\, \mathcal{A}_k^P(\vec{p}, h) \mathcal{A}_k^D(\vec{p}, h)\, e^{-S_k(\vec{p})} \exp\left[-iE_{\nu_k}(\vec{p})T + i\vec{p}\cdot\vec{L}\right]$$

$S_k(\vec{p}) = S_k^P(\vec{p}) + S_k^D(\vec{p})$ $e^{-S_k(\vec{p})}$ replaces energy-momentum δ-functions

$$S_k(\vec{p}) = \frac{(\vec{p}_P - \vec{p})^2}{4\,\sigma_{pP}^2} + \frac{\left[\left(E_P - E_{\nu_k}(\vec{p})\right) - (\vec{p}_P - \vec{p})\cdot\vec{v}_P\right]^2}{4\,\sigma_{pP}^2\,\lambda_P} + \frac{(\vec{p}_D - \vec{p})^2}{4\,\sigma_{pD}^2} + \frac{\left[\left(E_D - E_{\nu_k}(\vec{p})\right) - (\vec{p}_D - \vec{p})\cdot\vec{v}_D\right]^2}{4\,\sigma_{pD}^2\,\lambda_D}$$

Space-Time Transition Probability: $P_{\alpha\beta}(\vec{L},T) \propto |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2$

Transition Probability in Space: $P_{\alpha\beta}(\vec{L}) \propto \int \mathrm{d}T \, |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2$

For Ultrarelativistic Neutrinos:

$$P_{\alpha\beta}(\vec{L}) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2$$

$$+2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left[-2\pi i \, \frac{L}{L_{kj}^{\mathrm{osc}}} - \left(\frac{L}{L_{kj}^{\mathrm{coh}}} \right)^2 - 2\pi^2 \eta \left(\frac{\sigma_x}{L_{kj}^{\mathrm{osc}}} \right)^2 \right]$$

Oscillation Lengths: $L_{kj}^{\mathrm{osc}} = \dfrac{4\pi E}{|\Delta m_{kj}^2|}$ Coherence Lengths: $L_{kj}^{\mathrm{coh}} = \dfrac{4\sqrt{2\omega} E^2}{|\Delta m_{kj}^2|} \sigma_x$

η and ω depend on Production and Detection processes ($\eta \sim \omega \sim 1$)

Necessary Localization: $\sigma_x \ll L_{kj}^{\mathrm{osc}}$

Otherwise: neutrino mass measurement $\Longleftrightarrow$ no oscillations

$$\sigma_x \gtrsim L_{kj}^{\mathrm{osc}} \implies \delta m_k^2 \sim E \delta E \sim E \sigma_p \sim \frac{|\Delta m_{kj}^2| L_{kj}^{\mathrm{osc}}}{\sigma_x} \lesssim |\Delta m_{kj}^2|$$

Estimates of Coherence Length

$$L^{\text{osc}} = \frac{4\pi E}{|\Delta m^2|} = 2.5 \frac{(E/\text{MeV})}{(|\Delta m^2|/\text{eV}^2)} \text{m} \qquad L^{\text{coh}} = \frac{4\sqrt{2}\omega E^2}{|\Delta m^2|}\sigma_x \sim 10^{12} \frac{(E^2/\text{MeV}^2)}{(|\Delta m^2|/\text{eV}^2)} \left(\frac{\sigma_x}{\text{m}}\right) \text{m}$$

Process	$\lvert\Delta m^2\rvert$	L^{osc}	σ_x	L^{coh}
$\pi \to \mu + \nu$ at rest in vacuum: $E \simeq 30\,\text{MeV}$ natural linewidth	$2.5 \times 10^{-3}\,\text{eV}^2$	$30\,\text{km}$	$\tau_\pi \sim 10\,\text{m}$	$\sim 10^{16}\,\text{km}$
$\pi \to \mu + \nu$ at rest in matter: $E \simeq 30\,\text{MeV}$ collision broadening	$2.5 \times 10^{-3}\,\text{eV}^2$	$30\,\text{km}$	$\tau_{\text{col}} \sim 10^{-5}\,\text{m}$	$\sim 10^{10}\,\text{km}$
$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ at rest in matter: $E \leq 50\,\text{MeV}$ collision broadening	$1\,\text{eV}^2$	$\leq 125\,\text{m}$	$\tau_{\text{col}} \sim 10^{-10}\,\text{m}$	$\lesssim 10^2\,\text{km}$
${}^7\text{Be} + e^- \to {}^7\text{Li} + \nu_e$ in solar core: $E \simeq 0.86\,\text{MeV}$ collision broadening	$7 \times 10^{-5}\,\text{eV}^2$	$31\,\text{km}$	$\tau_{\text{col}} \sim 10^{-9}\,\text{m}$	$\sim 10^4\,\text{km}$

Conclusions

⇝ Standard expression for Oscillation Length of Ultrarelativistic Neutrinos is robust.

⇝ Wave Packet Treatment is necessary for $T \simeq L \Leftrightarrow$ Oscillations in Space.

⇝ Quantum Field Theoretical Wave Packet Models confirm Standard Oscillation Length and allows to calculate Coherence Length.

Neutrino Unbound

http://www.to.infn.it/~giunti/NU

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